Eur. Phys. J. C 17, 527–534 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000498

THE EUROPEAN PHYSICAL JOURNAL C

Quantum anti-de Sitter space

Z. Chang[1,2]

[1] Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 München, Germany
[2] Institute of High Energy Physics, Academia Sinica, P.O. Box 918(4), 100039 Beijing, P.R. China

Received: 14 April 1999 / Revised version: 13 December 1999 /
Published online: 25 September 2000 – © Springer-Verlag 2000

Abstract. The quantum anti-de Sitter (AdS) group and the quantum AdS space are discussed. A differential calculus on the quantum AdS space is introduced. A conjugation on the quantum AdS algebra is deduced from the corresponding conjugation on the quantum AdS group.

1 Introduction

The classical anti-de Sitter (AdS) space [1] is an Einstein space with maximal symmetry and positive cosmological constant. It is useful to consider the AdS space as a submanifold of a pseudo-Euclidean five-dimensional embedding space with Cartesian coordinates x^a and metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1, -1)$,

$$\eta_{ab} x^a x^b = -\frac{1}{a^2}.$$

With this definition of the AdS space, it is obvious that the symmetry group (isometry group) of the classical AdS space is $SO(3, 2)$. The group $SO(3, 2)$ plays the same role as the Poincaré group does on Minkowski space. In fact, the Poincaré group can be obtained from $SO(3, 2)$ by a contraction [2]. The AdS space has rather peculiar features [3].

The quantum field theory on the AdS space has been of interest for a long time. In the early 80's, there was great interest in four-dimensional N-extended supergravities for which the global $SO(N)$ is promoted to a gauge symmetry [4]. In these theories the underlying supersymmetry algebra is no longer Poincaré, but rather AdS. An important ingredient in these developments was that the AdS $\times$ S^7 geometry was not fed in by hand but resulted from a spontaneous compactification, i.e., the vacuum state was obtained by finding a stable solution of the higher-dimensional field equations. There has recently been a revival of interest in the AdS space brought about by the conjectured duality between string (or M) theory in the bulk of the AdS and a conformal field theory on the boundary of the AdS [5]. This is one of the most important points of progress in the so-called non-perturbative superstring theory. Many new results have been obtained by making use of this conjecture [6,7]. Here the AdS geometry plays a central role. It might seem very strange that quantum theories in different space-time dimensions could be equivalent. This possibility is related to the fact that the theory in the large dimension is (among other things) a quantum theory of gravity. For such a theory the concept of holography has been introduced as a generic property. The Maldacena conjecture is an example of a realization of the holography.

According to general relativity, gravity is nothing but space-time geometry. The poor understanding of physics at very short distances indicates that the small-scale structure of space-time might not be adequately described by classical continuum geometry. It has for a long time been suspected that the non-commutivity of space-time might be a realistic picture of how space-time behaves near the Planck scale, where strong quantum fluctuations of gravity may make points in space fuzzy [8]. It is known that the superstring theory may be the only possible candidate for a quantum theory of gravity [9]. The recent efforts to unify by non-perturbative dualities all the known five perturbative superstring theories is referred to as M theory [10]. Many physicists believe that the M theory is a fundamental quantum theory in eleven-dimensional space-time. The BFSS matrix model [11] was proposed for the microscopic description of the M theory in discrete light-cone quantization. The basic block in the matrix model is a set of N partons, called D_0-branes, on which strings can end. A novel feature of the M(atrix) theory is that the nine transverse coordinates of the D_0-branes are promoted into $N \times N$ Hermitian matrices. Thus the non-commutative geometry is a proper description for the superstring or M theory [12,13].

Bear in mind that the non-commutative geometry description is a strong candidate of realistic pictures of space-time behaviors at the Planck scale and the Maldacena conjecture is a concrete realization of the holography – a generic feature of a quantum theory of gravity; it is natural to think that the proper geometry of the Maldacena conjecture may be the non-commutative AdS space.

In this paper, we discuss the quantum AdS space and the quantum AdS group. A differential calculus on the quantum AdS space is presented. We give a realization of elements of the quantum group $SO_q(5)$ in terms of generators of the quantum algebra $U_q(so(5))$. Conjugations on the quantum algebra are deduced from corresponding conjugations on the quantum group by making use of this relationship between the quantum group and the quantum algebra.

This paper is organized as follows. In Sect. 2, we present preliminary knowledge about the quantum group theory and the differential calculus on the quantum orthogonal space. A concrete realization of the quantum algebra $U_q(so(5))$ in the quantum orthogonal space is shown in Sect. 3. In Sect. 4, conjugations on the quantum orthogonal group and the quantum orthogonal space are constructed to get the quantum AdS group and the quantum AdS space both for the $|q| = 1$ and the q real cases. We present a differential calculus on the quantum AdS space in Sect. 5. The reality of the differential calculus is discussed. Section 6 is devoted to the study of the quantum AdS algebra.

2 Differential calculus on quantum space

The quantum $\mathcal{R}$ matrix for the quantum group $SO_q(5)$ is of the form [14]

$$\begin{aligned}\mathcal{R} = q \sum_{\substack{i=-2\\ i\neq 0}}^{2} e_i^i \otimes e_i^i + \sum_{\substack{i,j=-2\\ i\neq j,j'\\ \text{or } i=j=0}}^{2} e_i^i \otimes e_j^j \\ + q^{-1} \sum_{\substack{i=-2\\ i\neq 0}}^{2} e_{i'}^{i'} \otimes e_i^i \\ + \lambda \left[\sum_{\substack{i,j=-2\\ i>j}}^{2} e_j^i \otimes e_i^j - \sum_{\substack{i,j=-2\\ i>j}}^{2} q^{\rho_i - \rho_j} e_j^i \otimes e_{j'}^{i'} \right],\end{aligned} \quad (1)$$

where $\lambda = q - q^{-1}$, $i' = -i$ and $\rho_i = (3/2, 1/2, 0, -1/2, -3/2)$.

By making use of the $\mathcal{R}$ matrix, the quantum group $SO_q(5)$ with entries

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \end{pmatrix}$$

can be written in the standard form

$$\mathcal{R} T_1 T_2 = T_2 T_1 \mathcal{R}, \quad (2)$$

where $T_1 \equiv T \otimes 1$ and $T_2 \equiv 1 \otimes T$.

The T matrix also should satisfy the orthogonal relations

$$T^t C T = C, \quad T C T^t = C, \quad (3)$$

where C_{ij} is the metric on the quantum orthogonal space,

$$C = \begin{pmatrix} & & & & q^{-3/2} \\ & & & q^{-1/2} & \\ & & 1 & & \\ & q^{1/2} & & & \\ q^{3/2} & & & & \end{pmatrix}.$$

The Hopf algebra structure of the quantum orthogonal group $SO_q(5)$ is

$$\begin{aligned} &\Delta(T) = T \dot{\otimes} T, \\ &\epsilon(T) = I, \\ &S(T) = C T^t C^{-1} = \\ &\begin{pmatrix} t_{55} & q^{-1}t_{45} & q^{-3/2}t_{35} & q^{-2}t_{25} & q^{-3}t_{15} \\ qt_{54} & t_{44} & q^{-1/2}t_{34} & q^{-1}t_{24} & q^{-2}t_{14} \\ q^{3/2}t_{53} & q^{1/2}t_{43} & t_{33} & q^{-1/2}t_{23} & q^{-3/2}t_{13} \\ q^2t_{52} & qt_{42} & q^{1/2}t_{32} & t_{22} & q^{-1}t_{12} \\ q^3t_{51} & q^2t_{41} & q^{3/2}t_{31} & qt_{21} & t_{11} \end{pmatrix}, \end{aligned} \quad (4)$$

where the operation $\dot{\otimes}$ between two matrices A and B is defined by

$$(A \dot{\otimes} B)_{ij} = A_{ik} \otimes B_{kj}.$$

To define a differential calculus on the non-commutative algebra generated by the coordinates $\mathbf{x}(= \{x^i\},\ i = -2, -1, 0, +1, +2)$, we write down the spectral resolution of the $\hat{\mathcal{R}}$ ($\equiv \mathcal{R}P$, and P is a permutation operator $P: A \otimes B = B \otimes A$),

$$\begin{aligned} \hat{\mathcal{R}} &= q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{-4}\mathcal{P}_1, \\ \hat{\mathcal{R}}^{-1} &= q^{-1}\mathcal{P}_S - q\mathcal{P}_A + q^4\mathcal{P}_1, \end{aligned} \quad (5)$$

where $\mathcal{P}_S$, $\mathcal{P}_A$ and $\mathcal{P}_1$ are projection operators which act on the tensor product $\mathbf{x} \otimes \mathbf{x}$ of the fundamental representation $\mathbf{x}$, and project into the symmetric, antisymmetric and singlet irreducible representations with dimension 14, 10 and 1, respectively. It is convenient to give a concrete representation of the projector,

$$\begin{aligned} \mathcal{P}_1 &= \frac{1-q^2}{(1-q^5)(1+q^3)} (C^{-1})^{ij} C_{kl} e_i^k \otimes e_j^l, \\ \mathcal{P}_A &= \frac{1}{q+q^{-1}} \left(-\hat{\mathcal{R}} + q - (q - q^{-4})\mathcal{P}_1 \right), \\ \mathcal{P}_S &= \frac{1}{q+q^{-1}} \left(\hat{\mathcal{R}} + q^{-1} - (q^{-1} + q^{-4})\mathcal{P}_1 \right). \end{aligned}$$

Analogous to the commutative case, we define the commutation relations of the quantum orthogonal space by requiring the vanishing of their antisymmetric products. Here the quantum antisymmetric products are given by the projector $\mathcal{P}_A$,

$$\mathcal{P}_A(\mathbf{x} \otimes \mathbf{x}) = 0. \quad (6)$$

In components, the commutation relations among the coordinates x^i is [14]

$$\begin{aligned} & x^i x^j = q x^j x^i, \quad \text{for } i < j \text{ and } i \neq -j, \\ & q x^{+2} x^{-2} - q^{-1} x^{-2} x^{+2} = \frac{q^{1/2} - q^{-1/2}}{q - 1 + q^{-1}} \frac{1}{a^2}, \\ & q x^{+1} x^{-1} - q^{-1} x^{-1} x^{+1} = (1 - q^2) x^{+2} x^{-2} \\ & \quad + q \frac{q^{1/2} - q^{-1/2}}{q - 1 + q^{-1}} \frac{1}{a^2}. \end{aligned} \tag{7}$$

Here we have used the constraint that the quantum length ($L \propto \mathbf{x}^t C \mathbf{x} = 1/a^2$) is invariant under the quantum orthogonal group transformations.

The exterior derivative d [15] is an operator which gives a mapping from the coordinates to the differentials

$$d: \; x^i \longrightarrow dx^i.$$

The derivative ∂_i can be introduced by

$$d = dx^i \partial_i = C_{ij} dx^i \partial^j.$$

For the differential calculus on the quantum orthogonal space [16], the following relations are satisfied:

$$\begin{aligned} & x^i dx^j = q \hat{R}^{ij}_{kl} dx^k x^l, \\ & \mathcal{P}_S(d\mathbf{x} \wedge d\mathbf{x}) = 0, \quad \mathcal{P}_1(d\mathbf{x} \wedge d\mathbf{x}) = 0, \\ & \partial^i x^j = (C^{-1})^{ij} + q(\hat{\mathcal{R}}^{-1})^{ij}_{kl} x^k \partial^l, \\ & \mathcal{P}_A{}^{ij}_{kl} \partial_j \partial_i = 0, \\ & \partial^i dx^j = q^{-1} \hat{\mathcal{R}}^{ij}_{kl} dx^k \partial^l, \\ & \partial^i d = q^{-2} d\partial^i - (q^{-2} - q^3) \frac{1 - q^2}{(1 - q^5)(1 + q^{-3})} \\ & \qquad \times dx^i C_{jk} \partial^j \partial^k. \end{aligned} \tag{8}$$

3 Realization of quantum algebra on quantum space

An action of an algebra A on a space V is a bilinear map,

$$\mathcal{A}: \; A \otimes V \longrightarrow V, \quad p \otimes x \longrightarrow p \cdot x,$$

such that

$$(pq) \cdot x = p \cdot (q \cdot x), \quad 1 \cdot x = 1.$$

We call V an A-module. In the same sense as co-multiplication is dual to multiplication, co-actions are dual to actions. The co-action of a co-algebra B on a space V is a linear map

$$\omega_B: \; V \longrightarrow B \otimes V,$$

such that

$$(\omega_B \otimes \mathrm{id})\omega_B = (\mathrm{id} \otimes \omega_B)\omega_B, \quad (\mathrm{id} \otimes \epsilon)\omega_B = \mathrm{id}.$$

The co-algebra is referred to as a co-module [17].

It is known that the quantum group co-acts on the quantum space. The quantum group is a co-module. As a dual Hopf algebra of the quantum group, the quantum algebra acts on the quantum space. In analogy to the classical case, there may be a realization of the quantum algebra on the quantum space. Here we present a realization of the quantum algebra $U_q(so(5))$ on the quantum orthogonal space.

For convenience, we introduce the dilatation operator S_m ($m \leq 2$) [18] by

$$\begin{aligned} S_m &= 1 + q\lambda E_m + q^{2m+1} \lambda^2 L_m \Delta_m, \\ E_m &= \sum_{j=-m}^{m} x^j \partial_j, \\ \Delta_m &= \sum_{j=1}^{m} q^{\rho_j} \partial_j \partial_{-j} + \frac{q}{1+q} \partial_0 \partial_0, \\ L_m &= \sum_{i=1}^{m} q^{\rho_i} x^{-i} x^i + \frac{q}{1+q} x^0 x^0. \end{aligned}$$

The dilatation operator satisfies

$$S_m x^k = q^2 x^k S_m, \quad S_m \partial_k = q^{-2} \partial_k S_m, \quad \text{for } |k| \leq m. \tag{9}$$

Using the notations

$$\begin{aligned} y^{-1} &= x^{-1} + q^{3/2} \lambda L_1 \partial_{+1}, \\ \delta_{-1} &= \partial_{-1} + q^{3/2} \lambda \Delta_1 x^{+1}, \\ y^{-2} &= x^{-2} + q^{5/2} \lambda L_2 \partial_{+2}, \\ \delta_{-2} &= \partial_{-2} + q^{5/2} \lambda \Delta_2 x^{+2}, \end{aligned}$$

we can construct a set of independent basis elements [19] on the quantum orthogonal space

$$\begin{aligned} \mathcal{X}^{-2} &= S_2^{-1/2} \mu_{+2}^{-1/2} y^{-2}, \\ \mathcal{D}_{-2} &= q^{-1} S_2^{-1/2} \mu_{+2}^{-1/2} \delta_{-2}, \\ \mathcal{X}^{-1} &= \mu_{+2}^{-1/2} S_1^{-1/2} \mu_{+1}^{-1/2} y^{-1}, \\ \mathcal{D}_{-1} &= \mu_{+2}^{-1/2} q^{-1} S_1^{-1/2} \mu_{+1}^{-1/2} \delta_{-1}, \\ \mathcal{X}^0 &= \mu_{+2}^{-1/2} \mu_{+1}^{-1/2} x^0, \\ \mathcal{D}_0 &= \mu_{+2}^{-1/2} \mu_{+1}^{-1/2} \partial_0, \\ \mathcal{X}^{+1} &= \mu_{+2}^{-1/2} x^{+1}, \\ \mathcal{D}_{+1} &= \mu_{+2}^{-1/2} \partial_{+1}, \\ \mathcal{X}^2 &= x^2, \\ \mathcal{D}_2 &= \partial_2, \end{aligned} \tag{10}$$

where $(\mu_{\pm i})^{\pm 1} = \mathcal{D}_{\pm i} \mathcal{X}^{\pm i} - \mathcal{X}^{\pm i} \mathcal{D}_{\pm i}$ and $\mu_0^{1/2} = \mathcal{D}_0 \mathcal{X}^0 - \mathcal{X}^0 \mathcal{D}_0$.
We note that the μ_is satisfy simple commutation relations with the independent bases $\mathcal{X}$ and $\mathcal{D}$,

$$[\mu_i, \mu_j] = 0, \quad \mu_i \mathcal{X}^j = \mathcal{X}^j \mu_i \cdot \begin{cases} q^2, & \text{for } i = j, \\ 1, & \text{for } i \neq j, \end{cases}$$

$$\mu_i \mathcal{D}^j = \mathcal{D}^j \mu_i \cdot \begin{cases} q^{-2}, & \text{for } i = j, \\ 1, & \text{for } i \neq j. \end{cases}$$

For the independent basis of coordinates and derivatives on the quantum space [18,19], it is not difficult to prove that

$$\begin{aligned}
&\mathcal{D}_{-2}\mathcal{X}^{-2} = 1 + q^{-2}\mathcal{X}^{-2}\mathcal{D}_{-2},\\
&\mathcal{D}_{-1}\mathcal{X}^{-1} = 1 + q^{-2}\mathcal{X}^{-1}\mathcal{D}_{-1},\\
&\mathcal{D}_{0}\mathcal{X}^{0} = 1 + q\mathcal{X}^{0}\mathcal{D}_{0},\\
&\mathcal{D}_{+1}\mathcal{X}^{+1} = 1 + q^{2}\mathcal{X}^{+1}\mathcal{D}_{+1},\\
&\mathcal{D}_{+2}\mathcal{X}^{+2} = 1 + q^{2}\mathcal{X}^{+2}\mathcal{D}_{+2},\\
&[\mathcal{D}_i,\mathcal{D}_j] = 0, \quad [\mathcal{X}^i,\mathcal{X}^j] = 0,\\
&\mathcal{D}_i\mathcal{X}^j = \mathcal{X}^j\mathcal{D}_i, \quad \text{for } i \neq j.
\end{aligned} \tag{11}$$

The quantum universal enveloping algebra [20] $U_q(so(5))$ is of the form

$$\begin{aligned}
&[H_1,H_2] = 0,\\
&[H_1,X_1^{\pm}] = \pm 2X_1^{\pm}, \quad [H_1,X_2^{\pm}] = \mp X_2^{\pm},\\
&[H_2,X_1^{\pm}] = \mp X_1^{\pm}, \quad [H_2,X_2^{\pm}] = \pm X_2^{\pm},\\
&[X_1^+,X_1^-] = \frac{q^{H_1}-q^{-H_1}}{q-q^{-1}},\\
&[X_2^+,X_2^-] = \frac{q^{H_2}-q^{-H_2}}{q-q^{-1}},\\
&X_2^{\pm}(X_1^{\pm})^2 - [2]_q X_1^{\pm}X_2^{\pm}X_1^{\pm} + (X_1^{\pm})^2X_2^{\pm} = 0,\\
&X_1^{\pm}(X_2^{\pm})^3 - [3]_{\sqrt{q}}X_2^{\pm}X_1^{\pm}(X_2^{\pm})^2 + [3]_{\sqrt{q}}(X_2^{\pm})^2X_1^{\pm}X_2^{\pm}\\
&\qquad - (X_2^{\pm})^3X_1^{\pm} = 0,
\end{aligned} \tag{12}$$

where $[m]_q = (q^m - q^{-m})/(q - q^{-1})$.

In terms of the set of independent basis elements $\mathcal{X}^i$, $\mathcal{D}_i$ on the quantum space, we can write down a realization (a similar, but different representation was given in [19]) of the quantum universal enveloping algebra $U_q(so(5))$ in the fundamental representation:

$$\begin{aligned}
q^{H_1} &= \left(\frac{\mu_{-2}\mu_{+1}}{\mu_{-1}\mu_{+2}}\right)^{1/2},\\
X_1^- &= \sqrt{q^{-1}\lambda}\left(\frac{\mu_{-1}\mu_{+2}}{\mu_{-2}\mu_{+1}}\right)^{1/4}\mu_{+2}^{-1/2}\\
&\quad\times\left((\mu_{-2}\mu_{+1})^{1/2}\mathcal{X}^{-1}\mathcal{D}_{-2} - q\mathcal{X}^{+2}\mathcal{D}_{+1}\right),\\
X_1^+ &= \sqrt{q^{-1}\lambda}\left(\frac{\mu_{-1}\mu_{+2}}{\mu_{-2}\mu_{+1}}\right)^{1/4}\mu_{+2}^{-1/2}\\
&\quad\times\left((\mu_{-2}\mu_{+1})^{1/2}\mathcal{X}^{-2}\mathcal{D}_{-1} - q\mathcal{X}^{+1}\mathcal{D}_{+2}\right),\\
q^{H_2} &= \left(\frac{\mu_{-1}}{\mu_{+1}}\right)^{1/2},\\
X_2^- &= \sqrt{q^{1/2}\lambda}(\mu_{-1}\mu_{+1})^{-1/4}\\
&\quad\times\left(q^{-3/2}\mu_{-1}^{1/2}\mathcal{X}^{0}\mathcal{D}_{-1} - \mathcal{X}^{+1}\mathcal{D}_{0}\right),\\
X_2^+ &= \sqrt{q^{1/2}\lambda}(\mu_{-1}\mu_{+1})^{-1/4}\\
&\quad\times\left(q^{1/2}\mu_{-1}^{1/2}\mathcal{X}^{-1}\mathcal{D}_{0} - \mathcal{X}^{0}\mathcal{D}_{+1}\right).
\end{aligned} \tag{13}$$

Thus, we have a natural action of the quantum algebra $U_q(so(5))$ on the quantum orthogonal space. This may be referred to as the quantum counterpart of the Lie algebra realized on the Euclidean space. In fact, in the case of $q = 1$, this realization recovers an ordinary realization of the Lie algebra.

4 Quantum AdS group and quantum AdS space

To study the quantum AdS group, real forms and $*$-conjugations of the quantum orthogonal group should be introduced. A $*$-structure[1] on a Hopf algebra A is an algebra anti-automorphism $(\eta ab)^* = \bar{\eta}b^*a^*$ ($\forall a,b \in A$, $\forall \eta \in C$), a co-algebra automorphism $\Delta \cdot * = (* \otimes *) \cdot \Delta$, $\epsilon \cdot * = \epsilon$ and an involution $*^2 = 1$.

It is known that [14], on the quantum orthogonal groups, conjugations can be defined. The first type is trivially $T^{\times} = T$. Compatibility with the Yang–Baxter equation requires $\overline{\mathcal{R}} = \mathcal{R}^{-1}$ and $\overline{C}C^t = 1$. This occurs only for $|q| = 1$. Then the CTT relations are invariant under the $*$-conjugation. The quantum orthogonal group co-acts on the quantum orthogonal space. Thus, conjugations on the quantum orthogonal group may induce associated conjugations on the quantum space. More precisely, a conjugation on the quantum space is compatible with a conjugation on its quantum symmetry group if the co-action $\omega(x^a) = T^a_b \otimes x^b$ satisfies $\omega(\mathbf{x}^{\times}) = T^{\times} \otimes \mathbf{x}^{\times} \equiv \delta^{\times}(\mathbf{x})$. The unique associated conjugation on the quantum space here is $(x^a)^{\times} = x^a$. Clearly, we cannot get the desired quantum AdS space by this conjugation on the quantum orthogonal space. Further structures should be added to the conjugation operation to arrive at the quantum AdS group and the quantum AdS space. For this aim, we introduce another operation on the quantum orthogonal group as [22]

$$T^{\dagger} = \mathcal{D}T\mathcal{D}^{-1}, \tag{14}$$

where the matrix $\mathcal{D}$ is of the form

$$\mathcal{D} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

It is easy to check that the $\mathcal{D}$ matrix is a special element of the quantum orthogonal group, i.e.,

$$\begin{aligned}
&\mathcal{R}_{12}\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_2\mathcal{D}_1\mathcal{R}_{12},\\
&\mathcal{D}^tC\mathcal{D} = C, \quad \mathcal{D}C\mathcal{D}^t = C.
\end{aligned}$$

Thus, the † operation is compatible with the Hopf algebra structure of the quantum orthogonal group,

$$\begin{aligned}
&\mathcal{R}_{12}T_1^{\dagger}T_2^{\dagger} = T_2^{\dagger}T_1^{\dagger}\mathcal{R}_{12},\\
&(T^{\dagger})^tCT^{\dagger} = C, \quad T^{\dagger}C(T^{\dagger})^t = C,\\
&\Delta(T^{\dagger}) = \mathcal{D}\Delta(T)\mathcal{D}^{-1} = T^{\dagger} \otimes T^{\dagger},\\
&\epsilon(T^{\dagger}) = \epsilon(T), \quad S(T^{\dagger}) = [S(T)]^{\dagger}.
\end{aligned}$$

[1] There are other possible definitions for conjugation on the quantum AdS group; see, for example [21]

We finally obtain the desired AdS quantum group by using the combined operation of $\times$ and $\dagger$, i.e., the conjugation $T^* \equiv T^{\times\dagger} = \mathcal{D}T\mathcal{D}^{-1}$. The induced conjugation on the quantum space is $\mathbf{x}^* \equiv \mathbf{x}^{\times\dagger} = \mathcal{D}\mathbf{x}$. To show that the conjugation really gives the quantum AdS group and the quantum AdS space, we should find a linear transformation $\mathbf{x} \longrightarrow \mathbf{x}' = M\mathbf{x}, T \longrightarrow T' = MTM^{-1}$ such that the new coordinates $\mathbf{x}'$ and T' are real and the new metric $C' = (M^{-1})^t CM^{-1}$ diagonal in the $q \longrightarrow 1$ limit, $C'|_{q=1} = \mathrm{diag}(1,-1,-1,-1,1)$. Here the M matrix is given by

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & & 1 \\ -1 & & & & 1 \\ & & \mathrm{i}\sqrt{2} & & \\ & -1 & & 1 & \\ & 1 & & 1 & \end{pmatrix}.$$

The second conjugation of the quantum orthogonal group given by FRT [14] is realized via the metric, i.e., $T^\star = C^tTC^t$. The condition on the quantum $\mathcal{R}$ matrix is $\overline{\mathcal{R}} = \mathcal{R}$, which happens for $q \in R$. Again the CTT relations are invariant under such a conjugation operation. The corresponding real form of the quantum orthogonal group $SO_q(5)$ is $SO_q(5;R)$. The induced conjugation on the quantum space is $\mathbf{x}^\star = C^t\mathbf{x}$. Again, we cannot get the desired quantum AdS group and the quantum AdS space by this conjugation alone. To obtain the quantum AdS group, we introduce the operation $\ddagger$ on the quantum orthogonal group $SO_q(5)$ by

$$T^\ddagger = \mathcal{N}T\mathcal{N}^{-1}, \tag{15}$$

where the $\mathcal{N}$ matrix is of the form

$$\mathcal{N} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}.$$

The $\mathcal{N}$ matrix is also a special element of the quantum orthogonal group, i.e.,

$$\begin{aligned} &\mathcal{R}_{12}\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}_2\mathcal{N}_1\mathcal{R}_{12}, \\ &\mathcal{N}^tC\mathcal{N} = C, \quad \mathcal{N}C\mathcal{N}^t = C. \end{aligned} \tag{16}$$

Thus, the $\ddagger$ operation is compatible with the Hopf algebra structure of the quantum group. We obtain the desired AdS quantum group by using a combined operation of the $\star$ and $\ddagger$, i.e., the conjugation $T^* \equiv T^{\star\ddagger} = \mathcal{N}C^tTC^t\mathcal{N}^{-1}$. The induced conjugation on the quantum space is $\mathbf{x}^* \equiv \mathbf{x}^{\star\ddagger} = C^t\mathcal{N}\mathbf{x}$.

Parallel to the case of $|q| = 1$, we should prove that the combined conjugation really gives the quantum AdS group and the quantum AdS space by trying to find a linear transformation $\mathbf{x} \longrightarrow \mathbf{x}' = N\mathbf{x}, T \longrightarrow T' = NTN^{-1}$ such that the new coordinates $\mathbf{x}'$ and T' are real and the new metric $C' = (N^{-1})^tCN^{-1}$ is diagonal in the $q \longrightarrow 1$ limit, $C'|_{q=1} = \mathrm{diag}(1,-1,-1,-1,1)$. In fact, we find such a matrix which satisfies all these requirements to be of the form

$$N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & & q^{3/2} \\ & -\mathrm{i} & & -\mathrm{i}q^{1/2} & \\ & & \mathrm{i}\sqrt{2} & & \\ & q^{-1/2} & & -1 & \\ \mathrm{i}q^{-3/2} & & & & -\mathrm{i} \end{pmatrix}.$$

5 A differential calculus on the quantum AdS space

To study differential calculus on the quantum AdS space, we should introduce conjugate derivatives. There are two different conjugations on the quantum AdS space; these correspond to q being a phase or real, respectively. We present a differential calculus on the quantum AdS space for the two separate cases.

In the case of q being a phase, we notice that the conjugation on x^i, $\hat{\mathcal{R}}$ and C is given by

$$x^{i\,*} = \mathcal{D}_{ij}x^j, \quad \overline{\hat{\mathcal{R}}^{uv}_{ij}} = \hat{\mathcal{R}}^{-1}{}^{vu}_{ji}, \quad \overline{C^{ij}} = C^{ji}. \tag{17}$$

Because the matrix $\mathcal{D}$ is a special element of the quantum orthogonal group, we can prove that

$$\hat{\mathcal{R}}^{-1}{}^{ik}_{ld}\mathcal{D}_{kn}\mathcal{D}_{sl}\mathcal{D}_{ip}\mathcal{D}_{td} = \hat{\mathcal{R}}^{-1}{}^{st}_{pn}.$$

The $\hat{\mathcal{R}}$ matrix has the following symmetry properties

$$C^{im}\hat{\mathcal{R}}^{jn}_{mk}C_{nl} = \hat{\mathcal{R}}^{-1}{}^{ij}_{kl} = C_{km}\hat{\mathcal{R}}^{mi}_{ln}C^{nj}, \tag{18}$$

and

$$\hat{\mathcal{R}}^{ij}_{kl} = \hat{\mathcal{R}}^{kl}_{ij}. \tag{19}$$

Using the above relations, we obtain the action of the conjugate derivatives on the quantum AdS space,

$$\hat{\partial}_m x^s = \delta^s_m + q\hat{\mathcal{R}}^{sl}_{mk}x^k\hat{\partial}_l, \tag{20}$$

where $\hat{\partial}_m \equiv -q^{-1}\mathcal{D}_{mk}C_{kv}\,\partial^{v\,*}$.

This shows that the action of the conjugate derivatives on the quantum AdS space is almost the same as that of the derivatives in the case of $|q| = 1$. In fact, we can represent the conjugate derivatives ∂^* linearly in terms of the derivatives ∂,

$$\partial_v{}^* = -qC_{vk}\mathcal{D}_{km}\partial_m. \tag{21}$$

From a very general consideration [15], we know that there are two types of consistent derivatives ∂_i and $\tilde{\partial}_i$, which satisfy the following relations, respectively:

$$\begin{aligned} &\partial_i x^j = \delta^j_i + q\hat{\mathcal{R}}^{jk}_{il}x^l\partial_k, \\ &\tilde{\partial}_k x^v = \delta^v_k + q^{-1}\,\hat{\mathcal{R}}^{-1}{}^{vi}_{kj}\,x^j\tilde{\partial}_i. \end{aligned} \tag{22}$$

The first one is just the one given in (8). We show that the second one is related with the conjugate derivatives in the case of $q \in R$.

In the following, we present actions of the conjugate derivatives on the quantum AdS space in the case of $q \in R$. To proceed, we first conjugate the first relation in (22) and invert it to find an expression for $\partial_i^* x^j$ in terms of $x^j \partial_i^*$ by using

$$x^{i\,*} = C_{ji} \mathcal{N}_{jk} x^k$$

and the symmetry properties of the $\hat{\mathcal{R}}$ matrix (18) and (19). The result is

$$\begin{aligned} -q^5 \mathcal{N}_{ip} C_{ia} C^{va}\ \partial^{v\,*} x^s &= q^4 C_{ia} C_{p\delta} \hat{\mathcal{R}}^{p\delta}_{al} \mathcal{N}_{sl} \mathcal{N}_{ip} \\ &-q^4\ \hat{\mathcal{R}}^{-1}{}^{ik}_{ld} \mathcal{N}_{kn} \mathcal{N}_{sl} \mathcal{N}_{ip} \mathcal{N}_{td} x^n C_{db} C^{mb} \mathcal{N}_{dt}\ \partial^{m\,*}. \quad (23) \end{aligned}$$

Making use of (16), we know that

$$\hat{\mathcal{R}}^{-1}{}^{ik}_{ld} \mathcal{N}_{kn} \mathcal{N}_{sl} \mathcal{N}_{ip} \mathcal{N}_{td} = \hat{\mathcal{R}}^{-1}{}^{st}_{pn}.$$

Then we get

$$\hat{\partial}_p x^s = \delta^s_p + q^{-1}\ \hat{\mathcal{R}}^{-1}{}^{st}_{pn} x^n \hat{\partial}_t, \qquad (24)$$

where

$$\hat{\partial}_p = -q^5 \mathcal{N}_{ip} C_{ia} C^{va}\ \partial^{v\,*}.$$

$\hat{\partial}_i$ (or equivalently ∂_i^*) can be expressed algebraically in terms of x^k and ∂_l [23]:

$$\hat{\partial}_k = q^3 S_2^{-1} [\Delta_2, x^k]. \qquad (25)$$

6 Quantum AdS Algebra

Quantum field theory is usually constructed based on representations of an algebra. To get a quantum field theory on the non-commutative geometry, we should investigate the properties of the quantum AdS algebra [24]. First of all, we should give conjugations on the quantum AdS algebra. It is well known that, in Lie group theory, there is an exponential correspondence between the group and the algebra. However, in general, there is no such a direct transformation from the quantum algebra to the quantum group, besides duality. To deduce a conjugation on the quantum algebra from a corresponding conjugation on the quantum group, we try to construct a concrete realization of elements of the quantum group in terms of generators of the quantum algebra [25].

It is convenient to introduce the operator L^+ by

$$L^+ = \begin{pmatrix} l^+_{11} & l^+_{12} & l^+_{13} & l^+_{14} & l^+_{15} \\ & l^+_{22} & l^+_{23} & l^+_{24} & l^+_{25} \\ & & l^+_{33} & l^+_{34} & l^+_{35} \\ & & & l^+_{44} & l^+_{45} \\ & & & & l^+_{55} \end{pmatrix}, \qquad (26)$$

where

$$\begin{aligned}
l^+_{11} &= q^{H_1+H_2}, \\
l^+_{12} &= \lambda q^{-1/2} X_1^+ q^{(H_1/2)+H_2}, \\
l^+_{13} &= \lambda q^{-1/2} \left(X_1^+ X_2^+ - q^{-1} X_2^+ X_1^+ \right) q^{(H_1+H_2)/2}, \\
l^+_{14} &= \lambda q^{-5/2} \left(-q X_1^+ (X_2^+)^2 \right. \\
&\quad \left. + (q+1) X_2^+ X_1^+ X_2^+ - (X_2^+)^2 X_1^+ \right) q^{H_1/2}, \\
l^+_{15} &= \frac{\lambda^2 q^{-2}}{1+q} \left((X_1^+)^2 (X_2^+)^2 \right. \\
&\quad - (q+1+q^{-1}) X_1^+ X_2^+ X_1^+ X_2^+ \\
&\quad + (q+1+q^{-1}) X_1^+ (X_2^+)^2 X_1^+ \\
&\quad - (q+1+q^{-1}) X_2^+ X_1^+ X_2^+ X_1^+ + X_2^+ (X_1^+)^2 X_2^+ \\
&\quad \left. + (X_2^+)^2 (X_1^+)^2 \right), \\
l^+_{22} &= q^{H_2}, \\
l^+_{23} &= \lambda q^{-1/2} X_2^+ q^{H_2/2}, \\
l^+_{24} &= -\frac{\lambda^2}{q(1+q)} (X_2^+)^2, \\
l^+_{25} &= \lambda q^{-5/2} \left(X_1^+ (X_2^+)^2 \right. \\
&\quad \left. - (1+q) X_2^+ X_1^+ X_2^+ + q (X_2^+)^2 X_1^+ \right) q^{-H_1/2}, \\
l^+_{33} &= 1, \\
l^+_{34} &= -\lambda q^{-1} X_2^+ q^{-H_2/2}, \\
l^+_{35} &= \lambda q^{-1} \left(-q^{-1} X_1^+ X_2^+ + X_2^+ X_1^+ \right) q^{-(H_1+H_2)/2}, \\
l^+_{44} &= q^{-H_2}, \\
l^+_{45} &= -\lambda q^{-1/2} X_1^+ q^{-(H_1/2+H_2)}, \\
l^+_{55} &= q^{-(H_1+H_2)}. \qquad (27)
\end{aligned}$$

In the same manner, we introduce another operator L^- by

$$L^- = \begin{pmatrix} l^-_{11} & & & & \\ l^-_{21} & l^-_{22} & & & \\ l^-_{31} & l^-_{32} & l^-_{33} & & \\ l^-_{41} & l^-_{42} & l^-_{43} & l^-_{44} & \\ l^-_{51} & l^-_{52} & l^-_{53} & l^-_{54} & l^-_{55} \end{pmatrix}, \qquad (28)$$

where

$$\begin{aligned}
l^-_{11} &= q^{-(H_1+H_2)}, \\
l^-_{21} &= -\lambda q^{1/2} X_1^- q^{-(H_1/2+H_2)}, \\
l^-_{31} &= \lambda q^{3/2} \left(X_1^- X_2^- - q^{-1} X_2^- X_1^- \right) q^{-(H_1+H_2)/2}, \\
l^-_{41} &= \lambda q^{1/2} \left(q X_1^- (X_2^-)^2 \right. \\
&\quad \left. - (q+1) X_2^- X_1^- X_2^- + (X_2^-)^2 X_1^- \right) q^{-H_1/2}, \\
l^-_{51} &= \frac{\lambda^2 q^2}{1+q} \left((X_1^-)^2 (X_2^-)^2 \right. \\
&\quad - (q+1+q^{-1}) X_1^- X_2^- X_1^- X_2^- \\
&\quad + (q+1+q^{-1}) X_1^- (X_2^-)^2 X_1^- \\
&\quad - (q+1+q^{-1}) X_2^- X_1^- X_2^- X_1^- + X_2^- (X_1^-)^2 X_2^- \\
&\quad \left. + (X_2^-)^2 (X_1^-)^2 \right), \\
l^-_{22} &= q^{-H_2}, \\
l^-_{32} &= -\lambda q^{1/2} X_2^- q^{-H_2/2}, \\
l^-_{42} &= -\frac{\lambda^2 q}{1+q} (X_2^-)^2,
\end{aligned}$$

$$
\begin{aligned}
l^-_{52} &= \lambda q^{1/2}\left(-X^-_1(X^-_2)^2 + (1+q)X^-_2X^-_1X^-_2\right.\\
&\quad \left.-\ q(X^-_2)^2X^-_1\right)q^{H_1/2},\\
l^-_{33} &= 1,\\
l^-_{43} &= \lambda X^-_2 q^{H_2/2},\\
l^-_{53} &= \lambda q\left(-q^{-1}X^-_1X^-_2 + X^-_2X^-_1\right)q^{(H_1+H_2)/2},\\
l^-_{44} &= q^{H_2},\\
l^-_{54} &= \lambda q^{1/2}X^-_1 q^{(H_1/2)+H_2},\\
l^-_{55} &= q^{H_1+H_2}.
\end{aligned}
\tag{29}
$$

By making use of the operators $L^\pm$, we can rewrite the quantum universal enveloping algebra $U_q(so(5))$ into a compact form:

$$
\begin{aligned}
&\mathcal{R}_{12}L^\pm_1L^\pm_2 = L^\pm_2L^\pm_1\mathcal{R}_{12},\\
&\mathcal{R}_{12}L^-_1L^+_2 = L^+_2L^-_1\mathcal{R}_{12},\\
&\Delta(L^\pm) = L^\pm\dot{\otimes}L^\pm, \quad \epsilon(L^\pm) = 1,\\
&S(L^+) = \begin{pmatrix}
l^+_{55} & q^{-1}l^+_{45} & q^{-3/2}l^+_{35} & q^{-2}l^+_{25} & q^{-3}l^+_{15}\\
 & l^+_{44} & q^{-1/2}l^+_{34} & q^{-1}l^+_{24} & q^{-2}l^+_{14}\\
 & & l^+_{33} & q^{-1/2}l^+_{23} & q^{-3/2}l^+_{13}\\
 & & & l^+_{22} & q^{-1}l^+_{21}\\
 & & & & l^+_{11}
\end{pmatrix},\\
&S(L^-) = \begin{pmatrix}
l^-_{55} & & & & \\
ql^-_{54} & l^-_{44} & & & \\
q^{3/2}l^-_{53} & q^{1/2}l^-_{43} & l^-_{33} & & \\
q^2l^-_{52} & ql^-_{42} & q^{1/2}l^-_{32} & l^-_{22} & \\
q^3l^-_{51} & q^2l^-_{41} & q^{3/2}l^-_{31} & ql^-_{21} & l^-_{11}
\end{pmatrix}.
\end{aligned}
\tag{30}
$$

We give here an explicit relation between elements of the quantum group and generators of the quantum universal enveloping algebra,

$$
T = L^+\dot{\otimes}L^-, \quad \text{or} \quad T = L^-\dot{\otimes}L^+. \tag{31}
$$

It is straightforward to verify that the matrix T defined above really satisfies all relations which a quantum group should satisfy. Let $P^\otimes$ be a transposition operator, for arbitrary matrices A and B, which satisfies

$$
P^\otimes:\ A\dot{\otimes}B = B\dot{\otimes}A.
$$

Then, a co-multiplication Δ for these tensor operators T can be introduced by

$$
\Delta = (\mathrm{id}\otimes P^\otimes\otimes\mathrm{id})\Delta_{L^\pm}\dot{\otimes}\Delta_{L^\mp}. \tag{32}
$$

It is easy to check that

$$
\begin{aligned}
\Delta(T) &= (\mathrm{id}\otimes P^\otimes\otimes\mathrm{id})\Delta_{L^\pm}\dot{\otimes}\Delta_{L^\mp}(L^\pm\dot{\otimes}L^\mp)\\
&= (\mathrm{id}\otimes P^\otimes\otimes\mathrm{id})(L^\pm\dot{\otimes}L^\pm\dot{\otimes}L^\mp\dot{\otimes}L^\mp)\\
&= (L^\pm\dot{\otimes}L^\mp)\dot{\otimes}(L^\pm\dot{\otimes}L^\mp)\\
&= T\dot{\otimes}T.
\end{aligned}
$$

Define a co-unit operator ϵ by

$$
\epsilon = \epsilon_{L^\pm}\otimes\epsilon_{L^\mp}.
$$

Then we have $\epsilon(T) = 1$. Finally, an antipode operator S is of the form

$$
S = P(S_{L^\mp}\otimes S_{L^\pm})P^\otimes.
$$

By making use of (30), it is a straightforward calculation to check that

$$
\begin{aligned}
S(T) &= S(L^\pm\dot{\otimes}L^\mp)\\
&= P(S_{L^\mp}\otimes S_{L^\pm})P^\otimes(L^\pm\dot{\otimes}L^\mp)\\
&= P(S_{L^\mp}(L^\mp)\otimes S_{L^\pm}(L^\pm))\\
&= CT^tC^{-1}.
\end{aligned}
$$

In the case of $|q| = 1$, a conjugation on the quantum AdS group is given by

$$
\begin{aligned}
T^* &= \mathcal{D}T\mathcal{D}^{-1}\\
&= \begin{pmatrix}
t_{11} & t_{12} & -t_{13} & t_{14} & t_{15}\\
t_{21} & t_{22} & -t_{23} & t_{24} & t_{25}\\
-t_{31} & -t_{32} & t_{33} & -t_{34} & -t_{35}\\
t_{41} & t_{42} & -t_{43} & t_{44} & t_{45}\\
t_{51} & t_{52} & -t_{53} & t_{54} & t_{55}
\end{pmatrix}.
\end{aligned}
\tag{33}
$$

From (31), we directly deduce the corresponding quantum AdS algebra to be

$$
\begin{aligned}
&{X^+_1}^* = X^+_1, \quad {X^-_1}^* = X^-_1, \quad H^*_1 = -H_1,\\
&{X^+_2}^* = -X^+_2, \quad {X^-_2}^* = -X^-_2, \quad H^*_2 = -H_2.
\end{aligned}
\tag{34}
$$

For the case of q real, we know from the previous section that a conjugation on the quantum AdS group is given by

$$
T^* = \mathcal{N}C^tT(C^{-1})^t\mathcal{N}^{-1} =
$$

$$
\begin{pmatrix}
t_{55} & -qt_{54} & -q^{3/2}t_{53} & -q^2t_{52} & q^3t_{51}\\
-q^{-1}t_{45} & t_{44} & q^{1/2}t_{43} & qt_{42} & -q^2t_{41}\\
-q^{-3/2}t_{35} & q^{-1/2}t_{34} & t_{33} & q^{1/2}t_{32} & -q^{3/2}t_{31}\\
-q^{-2}t_{25} & q^{-1}t_{24} & q^{-1/2}t_{23} & t_{22} & -qt_{21}\\
q^{-3}t_{15} & -q^{-2}t_{14} & -q^{-3/2}t_{13} & -q^{-1}t_{12} & t_{11}
\end{pmatrix}.
\tag{35}
$$

For $T = L^\pm\dot{\otimes}L^\mp$, the proper conjugation operation is given by

$$
T^* = {L^\mp}^*\dot{\otimes}{L^\pm}^*, \quad \left({L^\pm}^*\right)_{ij} \equiv \left(L^\pm_{ij}\right)^*. \tag{36}
$$

Comparing (35) and (36) gives the corresponding quantum AdS algebra,

$$
\begin{aligned}
&{X^+_1}^* = -q^2X^-_1, \quad {X^-_1}^* = -q^{-2}X^+_1, \quad H^*_1 = H_1,\\
&{X^+_2}^* = q^{3/2}X^-_2, \quad {X^-_2}^* = q^{-3/2}X^+_2, \quad H^*_2 = H_2.
\end{aligned}
\tag{37}
$$

Acknowledgements. I would like to thank Prof. J. Wess for introducing this problem to me and for enlightening discussions. I am grateful to B.L. Cerchiai and H. Steinacker for valuable discussions. The work was supported in part by the National Science Foundation of China under Grant 19625512.

References

1. J. Petersen, Int. J. Mod. Phys. A **14**, 3597 (1999)
2. J. Lukiersky, H. Ruegg, A. Nowicki, V. Tolstoy, Phys. Lett. B **264**, 331 (1991); J. Lukiersky, A. Nowicki, H. Ruegg, Phys. Lett. B **293**, 344 (1992)
3. C. Fronsdal, Rev. Mod. Phys. **37**, 221 (1965); Phys. Rev. D **10**, 589 (1974)
4. D.Z. Freedmen, A. Das, Phys. Lett. B **74**, 333 (1977); S.M. Christensen, M.J. Duff, G.W. Gibbons, M. Rocek, Phys. Rev. Lett. **45**, 161 (1980)
5. J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998)
6. S. Gubser, I. Klebanov, A. Polyakov, Phys. Lett. B **428**, 105 (1998)
7. E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998)
8. J. Wess, Int. J. Mod. Phys. A **12**, 4997 (1997)
9. M. Green, J. H. Schwarz, E. Witten, Superstring theory (2 volumes) (Cambridge University Press 1986)
10. J.H. Schwarz, Lectures on superstring and M theory dualities, Nucl. Phys. (Proc. Suppl.) B **55**, 1 (1997)
11. T. Banks, W. Fischler, S.H. Shenker, L. Susskind, Phys. Rev. D **55**, 5112 (1997)
12. A. Connes, M.R. Douglas, A. Schwarz, JHEP **9802**, 003 (1998)
13. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999)
14. L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Leningrad Math. J. **1**, 193 (1990)
15. J. Wess, B. Zumino, Nucl. Phys. (Proc. Suppl.) B **18**, 302 (1990)
16. U. Carow-Watamura, M. Schlieker, S. Watamura, Z. Phys. C **49**, 439 (1991)
17. P. Schupp, Quantum groups, non-commutative differential geometry and applications, hep-th/9312075
18. O. Ogievetsky, Lett. Math. Phys. **24**, 245 (1992)
19. G. Fiore, Commun. Math. Phys. **169**, 475 (1995)
20. Z. Chang, Physics Reports **262**, 137 (1995)
21. H. Steinacker, Quantum groups, roots of unity and particles on quantized anti-de Sitter space, hep-th/9705211
22. P. Aschieri, Real forms of quantum orthogonal quantum groups, q-Lorents groups in any dimension, preprint LBNL-41692 (1998), math.QA/9805120
23. O. Ogievetsky, B. Zumino, Lett. Math. Phys. **25**, 121 (1992)
24. H. Steinacker, Commun. Math. Phys. **192**, 687 (1998)
25. Z. Chang, H.Y. Guo, K. Wu, J. Phys. A **31**, L495 (1998)